Title	ON A NON-LOCAL EQUATION DESCRIBING THE RICCI FLOW(Variational Problems and Related Topics)
Author(s)	KAVALLARIS, NIKOS I.; SUZUKI, TAKASHI
Citation	数理解析研究所講究録 (2007), 1528: 1-11
Issue Date	2007-01
URL	http://hdl.handle.net/2433/58905
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
1528 巻 2007 年 1-11

ON A NON-LOCAL EQUATION DESCRIBING THE RICCI FLOW

NIKOS I. KAVALLARIS AND TAKASHI SUZUKI

ABSTRACT. A non-local parabolic equation desribing the normalized Ricci flow is studied. The equation applies on a two-dimensional compact Riemannian manifold Ω without boundary, e.g. flat torus $\mathbf{T}^2$, and contains a nonlinearity of the form $\lambda\left(e^u/\int_\Omega e^u dx - 1/|\Omega|\right)$. Global existence for every $\lambda > 0$ and convergence to a steady state for $0 < \lambda < 8\pi$, under some additional assumptions for the initial data, are proved.

Key Words: Non-local parabolic problems, Ricci flow.

2000 *Mathematics Subject Classification:* Primary 35B40, 35B45; Secondary 35Q72.

1. INTRODUCTION-BACKGROUND AND DERIVATION OF THE PROBLEM

If (Ω, g_0) is a compact Riemannian surface then the normalized Ricci flow describes the evolution in time of the metric $g = g(t)$ on Ω satisfying the initial condition $g(0) = g_0$. More precisely g is given as the solution of the problem

$$\frac{\partial g}{\partial t} = (r - R)\, g, \quad t > 0 \tag{1.1}$$

$$g(0) = g_0, \tag{1.2}$$

where $R = R(t)$ stands for the scalar curvature while $r = r(t)$ represents the average scalar curvature which is given by the form

$$r(t) = \frac{\int_\Omega R(t)\, d\mu_t}{\int_\Omega d\mu_t} \tag{1.3}$$

where $\mu = \mu_t$ is the volume element. Due to Gauss-Bonnet's theorem there holds

$$\int_\Omega R(t)\, d\mu_t = 4\pi\chi(\Omega) \tag{1.4}$$

where $\chi(\Omega)$ stands for the Euler characteristic of the surface Ω and is given as $\chi(\Omega) = 2 - 2k(\Omega)$ where $k(\Omega)$ is the genus of Ω, i.e. the number of holes existing in the surface Ω. Now by (1.3), taking also into account (1.4), we conclude that r is independent of the metric g and remains a constant in time since the volume is preserved along the Ricci flow.

Let now suppose that Ω is a two-dimensional surface with positive scalar curvature, then by virtue of (1.4) the hypothesis $R > 0$ implies that $k(\Omega) = 0$ and uniformization theorem guarantees that

$$\Omega = S^2 \quad \text{and} \quad g = e^w g_0,$$

for a smooth function w, where g_0 is the standard metric on the two dimensional sphere S^2. It is known, see Lemma 5.3 in [8], that the scalar curvatures R_g and R_0 corresponding to metrics g and g_0 respectivelly are related by

$$R_g = e^{-w}(-\Delta w + R_0), \tag{1.5}$$

where $\Delta = \Delta_{g_0}$. In view of (1.4)

$$\int_{S^2} R_g\, d\mu_g = 8\pi \tag{1.6}$$

and setting $dx = d\mu_{g_0}$ we obtain

$$r = \frac{8\pi}{\int_{S^2} d\mu_g} = \frac{8\pi}{\int_{S^2} e^w dx}. \tag{1.7}$$

Furhermore integrating (1.5) over S^2 we derive

$$|S^2|R_0 = 8\,\pi. \tag{1.8}$$

Now by pluging (1.5) into (1.1) and using (1.7), (1.8) we end up with the non-local equation

$$\frac{\partial e^w}{\partial t} = \Delta w + 8\pi\left(\frac{e^w}{\int_{S^2} e^w\,dx} - \frac{1}{|S^2|}\right) \quad x \in S^2,\ \ t > 0 \tag{1.9}$$

describing the normalized Ricci flow in the two-dimensional sphere S^2. Along with (1.9) the initial condition

$$w(x,0) = w_0(x) \quad x \in S^2 \tag{1.10}$$

is considered.

The first attempt to be studied the long-time behaviour of $g(t)$ was by Hamilton. He proved, see [13], using also some geometric arguments the following convergence result

$$g(t) \to g_\infty \quad \text{in} \quad C^\infty(S^2) \quad \text{as} \quad t \to \infty, \tag{1.11}$$

where g_∞ is a smooth metric on S^2 of constant curvature, under the hypothesis $R > 0$, which eventually removed by Chow, [7]. Hamilton's proof is very complicated since it involves some geometric arguments, like Harnack's inequality for the scalar curvature, along with monotonicity of an awkward geometric quantity called "entropy" and soliton solutions of the Ricci flow. Bartz et al, [4], gave a simpler proof of (1.11) working on the equivalent problem (1.9)-(1.10). Actually, they first proved the global-in-time existence of solutions of problem (1.1)-(1.2) and then the convergence result (1.11) based on a gradient estimate of the form

$$|\nabla_{S^2} w| \le C, \tag{1.12}$$

with C depending only on w_0. The proof of estimate (1.12) follows the lines of an argument exisiting in [23] and is based on the Harnack's inequality for solutions of the Yamade flow although a more elementary argument is used for the uniqueness of the asymptotic limit in [4].

Our aim is to study the global existence and long-time behaviour of the initial value non-local problem

$$\frac{\partial e^w}{\partial t} = \Delta w + \lambda\left(\frac{e^w}{\int_\Omega e^w\,dx} - \frac{1}{|\Omega|}\right) \quad x \in \Omega,\ \ t > 0 \tag{1.13}$$

$$w(x,0) = w_0(x) \quad x \in \Omega \tag{1.14}$$

whrere λ is a positive parameter and Ω is assumed to be a two-dimensional compact Remannian surface without boundary. Taking into account the above analysis, we might think of problem (1.13)-(1.14) as desribing the normalized Ricci flow in a more general Remannian surface than the two-dimensional sphere and coincides with (1.9)-(1.10) for $\lambda = 8\pi$.

Under the change of variables $u = \lambda e^w$ and $t = \lambda^{-1}\tau$ problem (1.13)-(1.14) is transformed to

$$u_\tau = \Delta \log u + u - \frac{1}{|\Omega|}\int_\Omega u\,dx, \ x \in \Omega,\ \ \tau > 0 \tag{1.15}$$

$$u(x,0) = u_0(x) = \lambda\, e^{w_0},\ x \in \Omega, \tag{1.16}$$

where

$$\int_\Omega u(x,\tau)dx = \lambda, \tag{1.17}$$

coming out by integration of equation (1.15) over Ω, see also next section.

In the next section we prove that the non-local perturbation term in (1.15) has a smoothing effect, in fact for every $0 < \lambda < \infty$ (1.17) permits the solution u of (1.15)-(1.16) to remain positive for every $0 < t < \infty$. Combining this result with an upper estimate which guarantees that u remains also bounded for every time, so $\log u$ term does, and we are able to prove the global-in-time existence of problem (1.15)-(1.16) and hence of the equivalent problem (1.13)-(1.14). Section 3 is devoted to the study of the stability of problem (1.15)-(1.16). More precisely, for every $0 < \lambda < 8\pi$ using the Luapunov functional of problem (1.13)-(1.14) we obtain a gradient estimate of the form (1.12) for w and taking advantage of the special structure of the problem we finally prove that w and hence u converges to a steady state.

2. Global Existence

In this section we study the global-in-time existence of the problem

$$u_t = \Delta \log u + u - \frac{1}{|\Omega|}\int_\Omega u\,dx, \quad x \in \Omega, \quad 0 < t < T_{max}, \tag{2.1}$$

$$u(x,0) = u_0(x), \; x \in \Omega. \tag{2.2}$$

For the initial data we assume that

$$u_0(x) > 0, \text{ i.e. } \min_\Omega u_0(x) \geq c > 0, \text{ with } u_0(x) \in L^\infty(\Omega). \tag{2.3}$$

Local existence of problem (2.1)-(2.3) can be proved using some classical parabolic estimates exisiting in [16].

By integrating equation (2.1) over Ω, taking also into account that Ω is compact manifold without boundary, we derive the total mass conservation condition

$$\int_\Omega u(x,t)\,dx = \int_\Omega u_0(x)\,dx = \lambda, \text{ for } 0 \leq t \leq T_{max}, \tag{2.4}$$

(in case $T_{max} = \infty$ (2.4) holds only for $0 \leq t < \infty$) hence finally problem (2.1)-(2.2) becomes

$$u_t = \Delta \log u + u - \frac{\lambda}{|\Omega|}, \quad x \in \Omega, \quad 0 < t < T_{max}, \tag{2.5}$$

$$u(x,0) = u_0(x), \; x \in \Omega, \tag{2.6}$$

where $\lambda > 0$ is the parameter of the problem.

To prove global-in-time existence for the solution of the problem (2.5)-(2.6) we use comparison techiques. First we set without a proof a compariosn result will be used a lot in the following. Actually, using the maximum principle holding in compact manifolds, see [2] page 130, it is not difficult to prove the following comparison result. For similar results, see also [1, 9].

Lemma 2.1. *Let Ω be a compact Riemannian and $w \in C(\Omega \times [0,T]) \cup C^{2,1}(\Omega \times (0,T))$, for some $0 < T < \infty$, be a classical solution of*

$$\begin{aligned} w_t &\geq \psi(x,t,w)\Delta w + f(w), \quad \text{in } \Omega \times (0,T) \\ w(x,0) &= w_0(x), \; x \in \Omega, \end{aligned}$$

and let $z \in C(\Omega \times [0,T]) \cup C^{2,1}(\Omega \times (0,T))$, be a classical solution of

$$\begin{aligned} z_t &\leq \psi(x,t,z)\Delta z + f(z), \quad \text{in } \Omega \times (0,T), \\ z(x,0) &= z_0(x), \; x \in \Omega, \end{aligned}$$

where $\psi \in C^2(\Omega \times [0,T] \times [-N,N])$, $N = \max(||w||_\infty, ||z||_\infty)$, $\psi \geq k > 0$ for some constant $k > 0$, and $f \in C^3(\mathbb{R})$. If $w_0(x) \geq z_0(x)$, then $w(x,t) \geq z(x,t)$ in $\Omega \times [0,T]$.

In the following we will need a Benilan-type estimate, i.e. an estimate of the form

$$\frac{u_t(x,t)}{u(x,t)} \leq g(t),$$

which is provided by the following.

Proposition 2.2. *Let $u \in C(\Omega \times [0,T]) \cup C^{2,1}(\Omega \times (0,T))$, for some $0 < T < \infty$, be a solution of (2.5)-(2.6) then u satisfies*

$$\frac{u_t(x,t)}{u(x,t)} \leq \frac{e^t}{e^t - 1} \quad \text{in } \Omega \times (0,T], \tag{2.7}$$

for every $\lambda > 0$. Moreover there exists a constant C_0 depending only on $||u_0(\cdot)||_\infty$ such that

$$0 < u(x,t) \leq C_0 e^t \quad \text{in } \Omega \times [0,T]. \tag{2.8}$$

Proof. Let $v = \log u$, then v satisfies

$$v_t = e^{-v}\Delta v + 1 - \frac{\lambda e^{-v}}{|\Omega|}, \text{ in } \Omega \times (0,T) \tag{2.9}$$

$$v(x,0) = v_0(x) = \log(u_0(x)), \; x \in \Omega. \tag{2.10}$$

Differentiating now equation (2.5) with respect to t we obtain

$$u_{tt} = \Delta\left(\frac{u_t}{u}\right) + u_t, \text{ in } \Omega \times (0,T),$$

or equivalently, since $u(x,t) > 0$ in $\Omega \times (0,T)$,

$$\frac{u_{tt}u - u_t^2}{u^2} = \frac{1}{u}\Delta\left(\frac{u_t}{u}\right) + \frac{u_t}{u} - \left(\frac{u_t}{u}\right)^2, \text{ in } \Omega \times (0,T),$$

hence $p = u_t/u$ satisfies the initial value problem

$$p_t = e^{-v}\Delta p + p - p^2, \text{ in } \Omega \times (0,T), \quad p(x,0) = 0, \quad x \in \Omega. \tag{2.11}$$

We consider

$$q(x,t) = 1 + \frac{1}{e^{t+C_\delta} - 1},$$

where C_δ is a constant to be selected properly below, then it is easily verified that $q(x,t)$ satisfies the equation of (2.11). By chossing

$$C_\delta = \log\left(1 + \frac{1}{|\,||p(\cdot,\delta)||_\infty - 1|}\right) \geq 0;$$

we derive that $q(x,0) = 1 + 1/(e^{C_\delta} - 1) \geq p(x,\delta)$ and in view of Lemma 2.1 we obtain

$$p(x,t+\delta) \leq q(x,t) = 1 + \frac{1}{e^{t+C_\delta} - 1} \leq 1 + \frac{1}{e^t - 1} \text{ in } \Omega \times (0,T].$$

Taking the limit as $\delta \to 0$, in the above relation, we get that

$$\frac{u_t(x,t)}{u(x,t)} = p(x,t) \leq \frac{e^t}{e^t - 1}, \text{ in } \Omega \times (0,T].$$

In order to obtain an estimate of the form (2.8) we try to construct an upper solution of problem (2.5)-(2.6) or equivalently of problem (2.9)-(2.10). First we note that the solution of the problem

$$V_t = e^{-V}\Delta V + 1, \text{ in } \Omega \times (0,T) \tag{2.12}$$

$$V(x,0) = v_0(x) = \log(u_0(x)), \; x \in \Omega. \tag{2.13}$$

is an upper solution to (2.9)-(2.10). Therefore, to obtain an estimate of the form (2.8) it is sufficient to construct an upper solution to problem (2.12)-(2.13). It is easily verified that $z(x,t) = \log(C_0 e^t)$, where $C_0 = ||u_0(\cdot)||_\infty$, is an upper solution to problem (2.12)-(2.13) and so an upper solution to (2.9)-(2.10). Hence

$$v(x,t) \leq \log(C_0 e^t) \text{ in } \Omega \times [0,T],$$

which implies estimate (2.8). □

Remark 2.3. From the definition of C_0 it is obvious that the constant C in (2.8) is independent of the parameter λ. Therefore, due to (2.8), which is a uniform estimate with respect to λ, we conclude that $u(x,t)$ remains bounded for every $0 < t < \infty$ and for any $\lambda > 0$, but this is not enough to permit us studying the long-time behaviour of u for any $\lambda > 0$, see also Remark 2.8.

Remark 2.4. Relation (2.7), implies that the function $u(x,t)/(e^t - 1)$ is (monotone) decreasing as time t increases to $T = T_{max}$. Indeed, using (2.7) we obtain

$$\left(\frac{u(x,t)}{(e^t - 1)}\right)_t = \frac{(u_t(x,t) - u(x,t)e^t/(e^t - 1))}{e^t - 1} \leq 0 \text{ in } \Omega \times (0,T]. \tag{2.14}$$

In the following we prove a monotonicity result with respect to the parameter λ, more precisely there holds.

Lemma 2.5. *The solution of problem (2.9)-(2.10) is decreasing with respect to λ.*

Proof. Let set $k(x,t) = -v_\lambda(x,t)$, then by differentiating problem (2.9)-(2.10) with respect to λ we obtain that k satisfies

$$k_t - e^{-v}\Delta k - \left(\Delta v + \frac{\lambda}{|\Omega|}\right)e^{-v}k = \frac{e^{-v}}{|\Omega|} \geq 0 \text{ in } \Omega \times (0,T)$$
$$k(x,0) = 0, \ \ x \in \Omega.$$

Since the function $\left(\Delta v + \frac{\lambda}{|\Omega|}\right)e^{-v}$ is bounded for a classical solution v, using the maximum principle, see [1, 21], we derive that $k \geq 0$ and so $v_\lambda \leq 0$ in $\Omega \times [0,T]$. □

The main result of this section is the following.

Theorem 2.6. *Problem (2.5)-(2.6) has a global-in-time (classical) solution $u \in C(\Omega \times [0,\infty)) \cup C^{2,1}(\Omega \times (0,\infty))$, i.e. $T_{max} = \infty$, for every $\lambda > 0$.*

Proof. Since (2.8) holds, in order to prove global-in-time existence of the solution $u(x,t)$, i.e. $T_{max} = T = \infty$, it is sufficient to show that

$$u(x,t) \geq C > 0 \ \text{ in } \Omega \quad \text{for any} \quad t > 0, \tag{2.15}$$

where the constant C might depend on time t.

We assume that (2.15) holds only in $[0,T)$ for some $T < \infty$ and we will draw a contradiction. In the following we proceed as in [14], but pointing out now that the continuity of $u(x,T)$ cannot be obtained by Dini's theorem. By virtue of Proposition 5.18 in [8] we obtain that

$$|w_t| \leq C_T \quad \text{in} \quad \Omega \times [0,T) \tag{2.16}$$

or taking also into account(2.8), the estimate

$$|u_t| \leq C_T' \quad \text{in} \quad \Omega \times [0,T). \tag{2.17}$$

Relation (2.17) first yields the existence of

$$u(x,T) = u(x,t) + \int_t^T u_t(x,s)ds, \quad t \in (0,T) \tag{2.18}$$

and then

$$|u(x,T) - u(x',T)| \leq |u(x,t) - u(x',t)| + C_T'(T-t).$$

Now by choosing $t_0(\epsilon) \in (0,T)$ such that $C(T - t_0) < \epsilon/2$ and using also the fact that $x \longmapsto u(x,t_0)$ is unifromly continuous in (compact surface) Ω, we finally obtain,

$$\text{for every} \quad \epsilon > 0 \quad \text{there exists} \quad \delta(\epsilon) > 0 \quad \text{s.t.} \quad |x - x'| < \delta \Rightarrow |u(x,T) - u(x',T)| < \epsilon,$$

thus $u(x,T) = \lim_{t\uparrow T} u(x,t)$ is (uniformly) continuous in Ω.

$$u(x,t) \geq \epsilon_1 > 0, \ \text{ in } \Omega \times [0,\delta_1]. \tag{2.19}$$

Also due to (2.3), (2.4) we have $\int_\Omega u(x,T)\,dx > 0$ and since $u \in C(\Omega \times [0,T])$, there exists $x_0 \in \Omega$ and constants $\epsilon_2 > 0$, $0 < \delta_2 < T - \delta_1$ such that $\overline{B_{\delta_2}(x_0)} \subset \Omega$ and

$$u(x,t) \geq \epsilon_2 > 0, \ \text{ in } \overline{B_{\delta_2}(x_0)} \times [T - \delta_2, T]. \tag{2.20}$$

Using again (2.14) we derive

$$u(x,t) \geq \frac{(e^t - 1)\,u(x, T-\delta_2)}{e^{T-\delta_2} - 1} \geq \frac{(e^{\delta_1} - 1)\epsilon_2}{e^{T-\delta_2} - 1} > 0 \ \text{ in } \overline{B_{\delta_2}(x_0)} \times [\delta_1, T - \delta_2]. \tag{2.21}$$

Combining now (2.19)-(2.21) we obtain

$$u(x,t) \geq \epsilon_3 > 0 \ \text{ in } \overline{B_{\delta_2}(x_0)} \times [0,T] \cup (\Omega \setminus B_{\delta_2}(x_0)) \times \{0\}, \tag{2.22}$$

where

$$\epsilon_3 = \epsilon_3(T) =: \min\left\{\epsilon_1, \epsilon_2, \frac{(e^{\delta_1} - 1)\epsilon_2}{e^{T-\delta_2} - 1}\right\} > 0.$$

Now we consider the problem

$$\Delta z + e^z - \frac{\lambda}{|\Omega|} = 0, \;\; x \in \Omega_{\delta_2, x_0} = \Omega \setminus B_{\delta_2}(x_0), \tag{2.23}$$

$$z = \log \epsilon_3, \;\; x \in \partial\Omega_{\delta_2, x_0} = \partial B_{\delta_2}(x_0). \tag{2.24}$$

Using maximum principle arguments we can obtain that problem (2.23)-(2.24) has, for every $\lambda > 0$, a minimal solution provided that ϵ_3 is sufficiently small. Also, using maximum principle, see for example Lemma 1 page 519 in [10], for $\psi = \log \epsilon_3 - z$ which satisfies the problem

$$-\Delta\psi + \rho(x)\psi = \frac{\lambda}{|\Omega|} - \epsilon_3 \geq 0, \;\; x \in \Omega_{\delta_2, x_0},$$

$$\psi = 0, \;\; x \in \partial\Omega_{\delta_2, x_0},$$

with $\rho(x) = -e^{\mu \log \epsilon_3 + (1-\mu) z(x)} \in L^\infty(\Omega_{\delta_2, x_0})$ and $\lambda \geq \epsilon_3 |\Omega|$ we derive that $\psi \geq 0$ or equivalently

$$z \leq \log \epsilon_3 \;\text{ in }\; \Omega_{\delta_2, x_0}. \tag{2.25}$$

Taking into account (2.22) and (2.25) we have

$$z_t - e^{-z}\Delta z - 1 + \frac{\lambda e^{-z}}{|\Omega|} = 0 = v_t - e^{-v}\Delta v - 1 + \frac{\lambda e^{-v}}{|\Omega|}, \;\; (x,t) \in \Omega_{\delta_2, x_0} \times [0,T],$$

$$z(x,t) \leq \log \epsilon_3 \leq v(x,t), \;\; (x,t) \in \partial\Omega_{\delta_2, x_0} \times [0,T]$$

$$z(x,0) \leq \log \epsilon_3 \leq v(x,0), \;\; x \in \Omega_{\delta_2, x_0}$$

for every $\lambda \geq \epsilon_3 |\Omega|$ and $\epsilon_3 > 0$ sufficiently small. Therefore in view of Lemma 2.5 we obtain that

$$v(x,t) \geq z(x) \geq m = \min_{\Omega_{\delta_2, x_0}} z(x) > -\infty \;\text{ in }\; \Omega_{\delta_2, x_0} \times [0,T],$$

or

$$u(x,t) \geq e^m > 0 \;\text{ in }\; \Omega_{\delta_2, x_0} \times [0,T], \tag{2.26}$$

for every $\lambda > 0$ and $0 < \epsilon_3$ sufficiently small.

Combining (2.22) and (2.26) we derive

$$u(x,t) \geq C = C(T) := \min\{\epsilon_3, e^m\} > 0 \;\text{ in }\; \Omega \times [0,T]. \tag{2.27}$$

Since now $u(x,T) > 0$ in Ω, by the same arguments as above we obtain a classical solution $\tilde{u}(x,t)$ but with initial data $u(x,T)$ in $\Omega \times [0,\delta]$ for some $\delta > 0$. Then by defining $u(x,t) = \tilde{u}(x, t-T)$ for $(x,t) \in \Omega \times [T, T+\delta]$ we extend $u(x,t)$ to a classical solution, with initial data $u_0(x)$, in $\Omega \times [T, T+\delta]$, but this contradicts the fact that $T = T_{max} < \infty$. This completes the proof. □

Remark 2.7. By relation (3.4) we conclude that the lower bound in (2.15) is not uniform with respect to time, i.e. the constant C depends on t.

Remark 2.8. Although by Theorem 2.6 we obtain global-in-time existence of problem (2.5)-(2.6) for every $\lambda > 0$, we can study the long-time behaviour of the corresponding solution only for $0 < \lambda < 8\pi$. This due to the fact that only for this range of λ we can obtain a uniform $H^1(\Omega)$–bound by Fontana-Moser's inequality, see also section 3.

3. Stability

In this section we study the stability of the corresponding to (2.5)-(2.6) steady-state problem. By making the substitution $u = \lambda e^w$, problem (2.5)-(2.6) is transfomed to

$$\frac{\partial e^w}{\partial \tau} = \Delta w + \lambda \left(\frac{e^w}{\int_\Omega e^w} - \frac{1}{|\Omega|} \right) \tag{3.1}$$

$$w(x,0) = \log \frac{u_0(x)}{\lambda}, \tag{3.2}$$

where also has been used the time-scaling $\tau = \lambda^{-1} t$ as well as that

$$\int_\Omega e^{w(x,t)}\, dx = \frac{1}{\lambda} \int_\Omega u(x,t)\, dx = 1 \tag{3.3}$$

(in the following, for the sake of simplicity we use t instead of τ); then the corresponding steady-state problem takes the form

$$\Delta \phi + \lambda \left(\frac{e^\phi}{\int_\Omega e^\phi\, dx} - \frac{1}{|\Omega|} \right) = 0. \tag{3.4}$$

We consider the functional

$$J_\lambda(w) = \frac{1}{2} ||\nabla w||_2^2 - \lambda \left\{ \log \int_\Omega e^w\, dx - \frac{1}{|\Omega|} \int_\Omega w\, dx \right\}.$$

Using the fact that Ω is compact Riemannian manifold it is easily seen that the semiflow defined by the solution of (3.1)-(3.2) is gradient-like in $X = H^1(\Omega)$ in the sence that

$$\int_0^t ||e^{w/2} w_t||_2^2\, ds = J_\lambda(w_0) - J_\lambda(w(x,t)) \text{ for every } t > 0, \tag{3.5}$$

i.e. $J_\lambda(w)$ is a Luapunov functional of this semiflow.

We also note that the functional $J_\lambda(w)$ can be written in the form

$$J_\lambda(w) = \frac{1}{2} ||\nabla (w - \hat{w})||_2^2 - \lambda \left\{ \log \int_\Omega e^{w - \hat{w}}\, dx \right\},$$

where $\hat{w}(t) = \hat{w} = \frac{1}{|\Omega|} \int_\Omega w(x,t)\, dx$. Applying Moser-Fontana's inequality, see [11], in the preceding relation we obtain due to (3.5)

$$J_\lambda(w_0) \geq J_\lambda(w) \geq \frac{1}{2} \left(1 - \frac{\lambda}{8\pi}\right) ||\nabla w||_2^2 + \lambda(\log|\Omega| - 1) \tag{3.6}$$

or

$$\frac{1}{2} \left(1 - \frac{\lambda}{8\pi}\right) ||\nabla w||_2^2 \leq J_\lambda(w_0) + \lambda(1 - \log|\Omega|).$$

The latter, for $0 < \lambda < 8\pi$, due to Poincare-Wirtinger's inequality, yields that

$$||w||_{H^1(\Omega)} \leq C = C(w_0, \lambda, |\Omega|) < \infty \tag{3.7}$$

and hence by (3.6) we obtain

$$J_\lambda(w) > -C. \tag{3.8}$$

Relation (3.5) implies

$$\int_0^t ||e^{w/2} w_t||_2^2\, ds \leq J_\lambda(w_0) - \hat{w}$$

and via (3.3), (3.7) and (3.8) we derive

$$\int_0^t ||e^{w/2} w_t||_2^2\, ds \leq C_1 < \infty,$$

which implies

$$\int_0^\infty ||e^{w/2} w_t||_2^2\, ds < \infty, \tag{3.9}$$

since the constant C_1 does not depend on time t.

Now for $1 < q < 2$ by Hölder's inequality we have

$$\int_\Omega e^{q\,w}|w_t|^q\,dx \le \left(\int_\Omega e^w w_t^2\,dx\right)^{q/2} \left(\int_\Omega e^{q\,w/(2-q)}\,dx\right)^{(2-q)/2}, \tag{3.10}$$

while using Gilbarg-Trudinger's inequality,[12], since (3.7) holds, along with Young's inequality we derive that

$$\int_\Omega e^{\beta\,w}dx \le C_3\,|\Omega| e^{\tilde\beta ||w||_{H^1}} < \infty \quad \text{for every} \quad \beta > 0 \tag{3.11}$$

and using (3.9)-(3.11) we end up with

$$\int_0^\infty \left\|\frac{\partial e^w}{\partial t}\right\|_q^2 ds < \infty. \tag{3.12}$$

Let now consider the ω–limit set for problem (3.1)-(3.2),

$$\omega(w_0) := \{\psi \in C^2(\Omega) : \text{ there exists } t_n \to \infty \text{ s.t. } ||w(\cdot, t_n; u_0) - \psi(\cdot)||_{C^2(\Omega)} \to 0\}$$

and setting

$$E := \left\{\phi \in C^2(\Omega) : \phi \text{ satisfies (3.4) and } \int_\Omega e^\phi = 1\right\},$$

then the following result holds.

Proposition 3.1. *For every $w_0 \in H^2(\Omega)$ and $0 < \lambda < 8\pi$ there holds $\omega(w_0) \neq \emptyset$ and $\omega(w_0) \subset E$.*

Proof. Due to (3.7) there exists a sequence $t_n \uparrow \infty$ with $t_{n+1} \ge t_n + \delta$, for some $\delta > 0$ (taking a subsequence if it is necessary) and $w_\infty \in H^1(\Omega)$ such that

$$w(\cdot, t_n) \to w_\infty(\cdot) \quad \text{as} \quad n \to \infty \quad \text{in} \quad H^1(\Omega). \tag{3.13}$$

Moreover due to (3.12) we have

$$\lim_{n\to\infty} \int_{t_n}^{t_n+\delta} \left\|\frac{\partial e^w}{\partial t}\right\|_q^2 ds = 0,$$

and so there should be some sequence $\tilde t_n \in (t_n, t_n + \delta)$ such that

$$\left\|\frac{\partial e^w(\cdot, \tilde t_n)}{\partial t}\right\|_q \to 0 \quad \text{as} \quad n \to \infty. \tag{3.14}$$

Relation (3.13), along with (3.11), yields

$$e^{w(\cdot,\tilde t_n)} \to e^{w_\infty(\cdot)} \quad \text{in} \quad L^1(\Omega) \quad \text{as} \quad n \to \infty \tag{3.15}$$

and

$$e^{w(\cdot,\tilde t_n)} \to e^{w_\infty(\cdot)} \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad n \to \infty \tag{3.16}$$

Going back to problem (3.1)-(3.2) we can prove that $||\Delta w(\cdot, \tilde t_n)||_q < \infty$. Indeed, using (3.11) and (3.14) we obtain via equation (3.1)

$$\left(\int_\Omega |\Delta w(x, \tilde t_n)|^q\,dx\right)^{1/q} \le \left(\int_\Omega \left|\frac{\partial e^{w(x,\tilde t_n)}}{\partial t}\right|^q dx\right)^{1/q} + \left(\int_\Omega \lambda^q \left|e^{w(x,\tilde t_n)} - \frac{1}{|\Omega|}\right|^q dx\right)^{1/q} < \infty, \tag{3.17}$$

where constant K is independent of n, recaling that $\int_\Omega e^{w(x,\tilde t_n)}\,dx = 1$, hence $w(\cdot, \tilde t_n) \in W^{2,q}(\Omega)$ for $1 < q < 2$. Using Morrey's embedding for compact manifolds, see Theorem 2.20 in [2], we derive that $w(\cdot, \tilde t_n) \in C^\gamma(\Omega)$ for some $0 < \gamma < 1$. Furthermore, via the parabolic regularity we obtain that $w(\cdot, t) \in C^{2+\gamma}(\Omega)$ for $t \in (\tilde t_n + \tau_1, \tilde t_n + \tau_2)$ and $||w||_{C^{2+\gamma,1+\gamma/2}(\Omega\times(\tilde t_n+\tau_1,\tilde t_n+\tau_2))} < K_1 < \infty$ for some $0 < \tau_1 < \tau_2$. Therefore there exists a sequence $\tau_n \in (\tilde t_n + \tau_1, \tilde t_n + \tau_2)$ such that

$$w(\cdot, \tau_n) \to w_\infty \quad \text{as} \quad n \to \infty \quad \text{in} \quad C^{2+\gamma}(\Omega).$$

Then passing through the sequence τ_n to the limit of (3.1), taking also into account (3.15)-(3.16), we derive that w_∞ is classical solution to problem (3.1), hence the desired result. $\square$

Remark 3.2. Using the center manifold theory we can show

$$\text{for any} \quad t_k \uparrow \infty \quad \text{there exists} \quad \{t'_k\} \subset \{t_k\} \quad \text{s.t.} \quad w(\cdot, t'_k) \to w_\infty \in E \quad \text{in} \quad C^{2+\theta}(\Omega), \ 0 < \theta < 1,$$

which implies the compactness of each orbit and hence $\omega(w_0)$ is a compact connected set.

Remark 3.3. The hypothesis $w_0 \in H^2(\Omega)$, via Sobolev's imbedding gaurantees that w_0 is bounded and so u_0 is, hence we have the sufficient regularity assumed in relation (2.3).

Using (2.16) we can prove that

$$\int_\Omega e^{qw}|w_t|^q dx \to 0 \quad \text{as} \quad t \to \infty \tag{3.18}$$

for $1 < q < 2$.

In fact (2.16) yields the estimate

$$w_t(x,t) \geq Ce^{rt} \quad \text{in} \quad \Omega \times [0, \infty) \tag{3.19}$$

where $r = \lambda / \int_\Omega e^w dx = \lambda$, see [8].

Differentiating (3.1) with respect to t, then taking the dual product with w_t yields that

$$\frac{d}{dt}\int_\Omega e^w w_t^2 dx + \int_\Omega |\nabla w_t|^2 dx = \lambda \int_\Omega e^w w_t^2 dx + \int_\Omega e^w w_t w_{tt} dx$$

and using again equation (3.1) we end up with

$$\frac{d}{dt}\int_\Omega e^w w_t^2 dx + 2\int_\Omega |\nabla w_t|^2 dx = 2\lambda \int_\Omega e^w w_t^2 dx - \int_\Omega e^w w_t^3 dx. \tag{3.20}$$

Relation (3.20) by virtue of (3.19) takes the form

$$\frac{d}{dt}\int_\Omega e^w w_t^2 dx \leq (2\lambda - Ce^{\lambda t})\int_\Omega e^w w_t^2 dx \leq -C_\delta \int_\Omega e^w w_t^2 dx, \quad t \geq \delta$$

for some postive constant C_δ depending on δ, which implies

$$\int_\Omega e^w w_t^2 dx \to 0 \quad \text{as} \quad t \to \infty,$$

and hence

$$\int_\Omega e^{qw}|w_t|^q dx \to 0 \quad \text{as} \quad t \to \infty$$

for $1 < q < 2$.

But relation (3.17) in view of (3.11) and (3.18) yields that $w(\cdot,t) \in H^q(\Omega), 1 < q < 2$, and due to Sobolev embedding for $N = 2$ we obtain $w(\cdot,t) \in L^\infty(\Omega)$. Therefore the positive orbit $\gamma^+(w_0)$ is uniformly bounded and in the case where the steady state set E is discrete we have that the time-dependent solution $w(x,t)$ tends to a steady-state solution, see also Remark 3.2. Hence the following holds.

Theorem 3.4. *For every $w_0 \in H^2(\Omega)$ satisfying (3.19) and $0 < \lambda < 8\pi$ the solution of (3.1)-(3.2) converges in $C^2(\Omega)$ to a steady state, i.e. a solution of problem (3.4), under the hypothesis that E is discrete.*

Considering now initial data w_0 which is an upper solution of the steady-state problem (3.4), i.e.

$$\Delta w_0 + \lambda\left(\frac{e^{w_0}}{\int_\Omega e^{w_0}\, dx} - \frac{1}{|\Omega|}\right) \leq 0 \tag{3.21}$$

we can prove that $w(x,t)$ converges towards to a steady state. In fact, under hypothesis (3.21) we can prove the following monotonicity result which is a key-result for the study of the asymptotic behaviour of $w(x,t)$.

Lemma 3.5. *The solution $w(x,t)$ of (3.1)-(3.2) is nonincreasing in time for every $x \in \Omega$.*

Proof. Differentiating equation (3.1) with respect to t, taking also into account (3.3), we derive

$$e^w w_t^2 + e^w w_{tt} = \Delta w_t + \lambda e^w w_t$$

or

$$\nu_t - e^{-w}\Delta\nu - \lambda\nu = -w_t^2 \leq 0 \tag{3.22}$$

for $\nu = w_t$. Due to (3.21) we also have that

$$\nu(x,0) = w_t(x,0) \leq 0. \tag{3.23}$$

Applying now the maximum principle, see [1], to problem (3.22)-(3.23) we derive the desired result. □

Now we are ready to prove the main result of this section.

Theorem 3.6. *For every $w_0 \in H^2(\Omega)$ satisfying (3.21) and $0 < \lambda < 8\pi$ the solution of (3.1)-(3.2) converges in $C^2(\Omega)$ to a steady state, i.e. a solution of problem (3.4).*

Proof. Following the same steps as in the proof of Proposition 3.1 we can find a sequence $t_n \to \infty$ such that

$$w(\cdot,t_n) \to w_\infty \quad \text{as} \quad n \to \infty \quad \text{in} \quad C^{2+\gamma}(\Omega)$$

where w_∞ is a steady-state solution. In view of Lemma 3.5 we conclude that

$$w(\cdot,t) \to w_\infty \quad \text{as} \quad t \to \infty \quad \text{pointwise in} \quad \Omega, \tag{3.24}$$

which implies that the orbit $\gamma^+(w_0)$ is uniformly bounded in $L^\infty(\Omega)$ and consequently the desired result, i.e.

$$w(\cdot,t) \to w_\infty \quad \text{as} \quad t \to \infty \quad \text{in} \quad C^2(\Omega).$$

Otherwise there should be a sequence $t_n \to \infty$ and $w_1 \in C^2(\Omega)$, $w_1 \neq w_\infty$, such that

$$w(\cdot,t_n) \to w_1 \quad \text{as} \quad n \to \infty \quad \text{in} \quad C^2(\Omega),$$

and hence

$$w(\cdot,t_n) \to w_1 \quad \text{as} \quad n \to \infty \quad \text{in} \quad L^\infty(\Omega),$$

which contradicts (3.24). □

Remark 3.7. For the two dimensional sphere $\Omega = S^2$, it is proven, [8, 6, 17], by using an Onofri-Hong type inequality, that problem (3.4) for $0 < \lambda < 8\pi$ has only the trivial solution in

$$\overset{\circ}{H}{}^1(\Omega) := \left\{\phi \in H^1(\Omega) : \int_\Omega \phi\, dx = 0\right\},$$

The same holds for two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/a\mathbb{Z} \times b\mathbb{Z}$ where $\frac{b}{a} \geq \frac{2}{\pi}$, see [18], again for the parameter-range $(0, 8\pi)$. Therefore, in view of Theorem 3.4 we derive

$$w(\cdot,t) \to 0 \quad \text{as} \quad t \to \infty \quad \text{uniformly in} \quad \overset{\circ}{H}{}^1(\Omega),$$

for $\Omega = S^2, \mathbb{T}^2$.

Acknowledgements: N.Kavallaris would like to give sincere thanks to "The 21st COE Program Towards a New Basic Science : Depth and Synthesis" for a financial support for his attendance to this conference held in Kyoto.

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DIVISION OF MATHEMATICAL SCIENCE,
DEPARTMENT OF SYSTEM INNOVATION,
GRADUATE SCHOOL OF ENGINEERING SCIENCE,
OSAKA UNIVERSITY,
MACHIKANEYAMACHO 1-3,
TOYONAKASHI, 560-8531, JAPAN
E-mail address: nkaval@sigmath.es.osaka-u.ac.jp

DIVISION OF MATHEMATICAL SCIENCE,
DEPARTMENT OF SYSTEM INNOVATION,
GRADUATE SCHOOL OF ENGINEERING SCIENCE,
OSAKA UNIVERSITY,
MACHIKANEYAMACHO 1-3,
TOYONAKASHI, 560-8531, JAPAN
E-mail address: suzuki@sigmath.es.osaka-u.ac.jp